2020

MATHEMATICS

[GENERAL]

Paper : I

Full Marks : 100 Time : 3 Hours

The figures in the right-hand margin indicate marks.
Symbols, notations have their usual meanings.

GROUP-A

(Differential Calculus)

[Marks : 50]

1. Answer any **four** questions: $1\times4=4$

   a) Test the differentiability of

   $$f(x)=\begin{cases} x+1 & , \quad 0\le x\le 1 \\ 3-x & , \quad 1\le x\le 2, \end{cases}$$

   at x=1.

   b) State Lagrange's MVT.

   c) Find the radius of curvature of $\sqrt{x}+\sqrt{y}=1$ at $\left(\frac{1}{4},\frac{1}{4}\right)$.

   d) A monotone sequence is always convergent – Justify.

   e) Test the convergence of the series

   $$\frac{1}{3}+\left(\frac{2}{5}\right)^2+\left(\frac{3}{7}\right)^3+\dots+\left(\frac{n}{2n+1}\right)^n+\dots.$$

   f) Evaluate $\lim_{x\to0}(1+2x)^{\frac{x+3}{x}}$.

2. Answer any **six** questions: $2\times6=12$

   a) Show that $\log_e(1+x)<x-\frac{x^2}{2(1+x)}$ for $x>0$.

   b) If $f(x, y)=\begin{cases} xy & \text{when } |x|\ge|y| \\ -xy & \text{when } |x|<|y| \end{cases}$, show that $f_{xy}(0, 0)\ne f_{yx}(0, 0)$.

   c) If $\sum_{n=1}^{\infty}a_n$ is a convergent series of positive real numbers, will the series $\sum_{n=1}^{\infty}a_{2n}$ be convergent? Justify your answer.

   d) Evaluate $\lim_{x\to0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x}}$.

   e) Investigate for continuity at $(1, 2)$ of

   $$f(x, y)=\begin{cases} x^2+2y & , \quad (x, y)\ne(1, 2) \\ 0 & , \quad (x, y)=(1, 2) \end{cases}.$$

f) Find the value of x for which $(\sin x - \cos x)$ is a maximum or a minimum.

g) Find the pedal equation of the asteroid

$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}.$$

h) If $f(x, y) = |x| + |y|$, show that f is not differentiable at $(0, 0)$.

3. Answer any **four** questions: 6×4=24

a) i) Find all the asymptotes of the curve

$$y = \frac{3x}{2}\log\left(e - \frac{1}{3x}\right).$$

ii) Find the envelope of the family of lines $\frac{x}{a} + \frac{y}{b} = 1$, where the parameters are connected by $a^2 + b^2 = c^2$. (c being a given constant) 3+3=6

b) i) Test the convergence of the series

$$x + \frac{2^2x^2}{2!} + \frac{3^3x^3}{3!} + \frac{4^4x^4}{4!} + \ldots, \; x > 0.$$

ii) If $y = e^{m\sin^{-1}x}$, show that

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0.$$

3+3=6

c) i) If $f(x, y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2} & , \; (x, y) \neq (0, 0) \\ 0 & , \; (x, y) = (0, 0) \end{cases}$

show that $f_{xy}(0, 0) \neq f_{yx}(0, 0)$.

ii) If $f(x) = \begin{cases} 1 & , \; x \text{ is rational} \\ 0 & , \; x \text{ is irrational}, \end{cases}$

prove that $\lim_{x \to a} f(x)$ does not exist for any real number a. 4+2=6

d) i) Suppose that $x_n \to l$ and $y_n \to m$ as $n \to \infty$, then prove that $x_n + y_n \to l + m$ as $n \to \infty$.

ii) Prove that the sequence $\{x_n\}$ defined by $x_1 = \sqrt{2}$, $x_{n+1} = \sqrt{2x_n}$, for $n \geq 1$ converges to 2. 2+4=6

e) i) Show that for $y = x^3 \log x$,

$$\frac{d^n y}{dx^n} = (-1)^n \frac{6\lfloor n-4}{x^{n-3}}.$$

ii) Find the maximum and minimum values of $f(x) = a\sin^2 x + b\cos^2 x$, where $a > b$.

3+3=6

f) i) State and prove Eulers theoem for homogeneous function in two variable x, y of degree n.

ii) If $f(0)=0,\ f'(x)=\dfrac{1}{1+x^2}$, prove without the method of integration that

$$f(x)+f(y)=f\left(\frac{x+y}{1-xy}\right). \qquad 3+3=6$$

4. Answer any **one** question: 10×1=10

a) i) Show that the envelope of a family of circles whose centres lie on the rectangular hyperbola $xy=c^2$ and which pass through the centre of the hyperbola is $\left(x^2+y^2\right)^2=16c^2xy$.

ii) If ρ_1 and ρ_2 be the radii of curvature at the ends of conjugate diameters of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, prove that

$$\rho_1^{\frac{2}{3}}+\rho_2^{\frac{2}{3}}=\frac{\left(a^2+b^2\right)}{(ab)^{\frac{2}{3}}}. \qquad 5+5$$

b) i) If $u=ax^2+2hxy+by^2$, show that

$$\left(\frac{\partial u}{\partial x}\right)^2\frac{\partial^2 u}{\partial x^2}-2\frac{\partial u}{\partial x}.\frac{\partial u}{\partial y}.\frac{\partial^2 u}{\partial x\partial y}+\left(\frac{\partial u}{\partial y}\right)^2\frac{\partial^2 u}{\partial y^2}$$
$$=8\left(ab-h^2\right)u.$$

ii) If

$$f(x,y)=\begin{cases}\left(x^2+y^2\right)\log\left(x^2+y^2\right) & ,\ x^2+y^2\neq 0\\ 0 & ,\ x^2+y^2=0\end{cases}$$

show that $f_{xy}(0,0)=f_{yx}(0,0)$, although neither f_{xy} nor f_{yx} is continuous at $(0,0)$.

5+5

GROUP-B

(Integral Calculus)

[Marks : 30]

5. Answer any **four** questions: 2×4=8

a) Evalute $\int_0^2 |1-x|\,dx$.

b) Evaluate $\int_0^1 dy\int_0^1 f(x,y)\,dx$, where

$$f(x,y)=\begin{cases}\dfrac{1}{2} & ,\ y \text{ rational}\\ x & ,\ y \text{ irrational}.\end{cases}$$

c) Find the length of the circumference of the circle $x^2+y^2=25$.

d) Evaluate $\int_{-2}^{2}\dfrac{x^2\sin x}{x^6+12}\,dx$.

e) Find the value of $\int_0^{\frac{1}{2}} \frac{\sin^{-1} x\, dx}{\sqrt{1-x^2}}$.

f) Show that $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

6. Answer any **two** questions: 6×2=12

a) Show that:

i) $\int_0^{\infty} e^{-4x} x^{\frac{3}{2}} dx = \frac{3}{128}\sqrt{\pi}$.

ii) $\Gamma\left(\frac{1}{9}\right)\Gamma\left(\frac{2}{9}\right) \dots \Gamma\left(\frac{8}{9}\right) = \frac{3}{16}\pi^4$. 3+3=6

b) i) Evaluate $\iint_R \sin(x+y)\, dx\, dy$, where

$$R = \left\{0 \le x \le \frac{\pi}{2},\ 0 \le y \le \frac{\pi}{2}\right\}.$$

ii) Find the volume of the solid generated by revolving one arch of the cycloid $x = a(\theta - \sin\theta),\ y = a(1-\cos\theta)$ about its base. 3+3=6

c) i) Obtain a reduction formula for $\int_0^{\frac{\pi}{4}} \tan^n x\, dx$, n being a positive integer ≥ 1 and hence evaluate $\int_0^{\frac{\pi}{4}} \tan^6 x\, dx$.

ii) Evaluate $\int_0^{\frac{\pi}{2}} \sin^4 x \cos^8 x\, dx$. 3+3=6

7. Answer any **one** question: 10×1=10

a) i) Show that $\frac{1}{2} < \int_0^1 \frac{dx}{\sqrt{4 - x^2 + x^3}} < \frac{\pi}{6}$.

ii) Prove that $\iiint (x^2 + y^2 + z^2) xyz\, dx\, dy\, dz$ taken throughout the sphere $x^2 + y^2 + z^2 \le 1$ is zero. 5+5=10

b) i) Find the area bounded by $y = 6 + 4x - x^2$ and the chord joining (–2, –6) and (4, 6).

ii) Show that the arc of the upper half of the cardiode $r = a(1 - \cos\theta)$ is bisected at $\theta = \frac{2}{3}\pi$. Also show that the perimeter of the curve is 8a. 5+5=10

GROUP–C
(Differential Equations)
(Marks : 20)

8. Answer any **two** questions: 1×2=2

a) Find the order and degree of the differential equation

$$\left(\frac{d^2y}{dx^2}\right)^2 + \frac{d^2y}{dx^2} + x \sin y = 0.$$

b) Construct a differential equation by the elimination of the arbitrary constants a and b from the equation $ax^2 + by^2 = 1$.

c) Find an integrating factor of the differential equation

$$\left(y^2 + 2x^2y\right)dx + \left(2x^3 - xy\right)dy = 0.$$

9. Answer any **one** question: 2×1=2

a) Find the Particular Integral (P.I.) of the differential equation $\left(D^2 - 5D - 6\right)y = e^{4x}$.

b) Solve $\left(\cos y + y\cos x\right)dx + \left(\sin x - x\sin y\right)dy = 0$.

10. Answer any **one** question: 6×1=6

a) Find the general and singular solutions of

$$16x^2 + 2p^2y - p^3x = 0. \left(p = \frac{dy}{dx}\right).$$

b) Prove that the necessary and sufficient condition that the differential equation

$$M(x, y)dx + N(x, y)dy = 0$$

be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

11. Answer any **one** question: 10×1=10

a) i) Find the curve for which the product of the intercepts of the tangent line on the co-ordinate axes is equal to a.

ii) The acceleration of a moving particle being proportional to the cube of its velocity and negative, show that the distance passed over in time t is given by

$$s = \frac{\left\{\sqrt{2kv_0^2t + 1} - 1\right\}}{kv_0},$$

v_0 being the initial velocity and the distance is measured from the position of the particle at time $t = 0$. 10

b) i) Solve: $\frac{d^2y}{dx^2} + a^2y = \sec ax$.

ii) Solve: $\frac{dx}{dt} + 5x + y = e^t$

$\frac{dy}{dt} + 3y - x = e^{2t}$ 5+5
